MATH 2050B 2017-18

Mathematical Analysis I

Tutorial Notes

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1 The Real Numbers

1.1 Axioms of Real Numbers

(A1) $a + b = b + a, \forall\, a, b \in \mathbb{R}$,

(A2) $(a + b) + c = (a + (b + c)), \forall\, a, b, c \in \mathbb{R}$,

(A3) $\exists\, 0 \in \mathbb{R}$, s.t. $0 + a = a = a + 0\ \forall\, a \in \mathbb{R}$,

(A4) $\forall\, a \in \mathbb{R}, \exists\, b \in \mathbb{R}$, s.t. $a + b = 0 = b + a$. Then we denote this b as $-a$,

(M1) $a \cdot b = b \cdot a\ \forall\, a, b \in \mathbb{R}$.

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)\ \forall\, a, b, c \in \mathbb{R}$,

(M3) $\exists\, 1 \in \mathbb{R}$, s.t. $1 \cdot a = a = a \cdot 1\ \forall\, a \in \mathbb{R}$,

(M4) $\forall\, a \in \mathbb{R} \setminus \{0\},\ \exists\, b \in \mathbb{R}$, s.t. $a \cdot b = 1 = b \cdot a$. Then we denote this b as $\frac{1}{a}$,

(D1) $a \cdot (b + c) = a \cdot b + a \cdot c\ \forall\, a, b, c \in \mathbb{R}$,

(D2) $0 \neq 1$,

(O1) Given $a, b \in \mathbb{R}$, there are one and the only one of the following case will occur:

- $a = b$ • $a < b$ • $a > b$

(O2) if $a > b$ for some $a, b \in \mathbb{R}$, then $a + c > b + c\ \forall\, c \in \mathbb{R}$,

(O3) if $a > b$ for some $a, b \in \mathbb{R}$, then $ac > bc\ \forall\, c > 0$,

(O4) if $a > b$ and $b > c$ for some $a, b, c \in \mathbb{R}$, then $a > c$.

(Completeness) Every bounded above nonempty subset in $\mathbb{R}$ has a Supremum in $\mathbb{R}$.

1.2 Properties of Real Numbers

(i) 0, 1 are unique, $-a$ is unique for each $a \in \mathbb{R}$, $\frac{1}{a}$ is unique for each $a \in \mathbb{R} \setminus \{0\}$,

(ii) if $a + c = b + c$ for some $a, b, c \in \mathbb{R}$, then $a = b$.

(iii) $a \cdot 0 = 0\ \forall\, a \in \mathbb{R}$,

(iv) $-a = (-1) \cdot a\ \forall\, a \in \mathbb{R}$,

(v) $-(-a) = a\ \forall\, a \in \mathbb{R}$,

(vi) $(-a)(-b) = a \cdot b\ \forall\, a, b \in \mathbb{R}$,

(vii) if $a > b$ for some $a, b \in \mathbb{R}$, then $-a < -b$,

(viii) if $a > b$ for some $a, b \in \mathbb{R}$, then $ca < cb\ \forall\, c < 0$,

(ix) $a^2 := a \cdot a > 0\ \forall\, a \in \mathbb{R} \setminus \{0\}$.

(x) $1 > 0$,

(xi) $2 > 1 > \frac{1}{2} > 0$,

(xii) if $a \in \mathbb{R}$ satisfies $0 \leq a < \varepsilon\ \forall\, \varepsilon > 0$, then $a = 0$.

Proof

(i) Suppose $0' \in \mathbb{R}$ also satisfies (A3), then by (A3) of 0 and $0'$,we have $0 = 0 + 0' = 0'$.

The other cases are similar, so I left them as exercise.

(ii) Note that

$$\begin{aligned} a &\overset{(A3)}{=} a + 0 \\ &\overset{(A4)}{=} a + [c + (-c)] \\ &\overset{(A2)}{=} (a + c) + (-c) \\ &\overset{\text{assumption}}{=} (b + c) + (-c) \\ &\overset{(A2)}{=} b + [c + (-c)] \\ &\overset{(A4)}{=} b + 0 \\ &\overset{(A3)}{=} b. \end{aligned}$$

(iii) Note that $0 + a \cdot 0 \overset{(A3)}{=} a \cdot 0 \overset{(A3)}{=} a \cdot (0 + 0) \overset{(D1)}{=} a \cdot 0 + a \cdot 0$, by (ii), we have $a \cdot 0 = 0$.

(iv) Note that

$$\begin{aligned} (-1) \cdot a &\overset{(A3)}{=} (-1) \cdot a + 0 \\ &\overset{(A4),(A2)}{=} [(-1) \cdot a + a] + (-a) \\ &\overset{(M3)}{=} [(-1) \cdot a + 1 \cdot a] + (-a) \\ &\overset{(D1)}{=} (-1 + 1) \cdot a + (-a) \\ &\overset{(A4)}{=} 0 \cdot a + (-a) \\ &\overset{\text{(iii)}}{=} 0 + (-a) \\ &\overset{(A3)}{=} -a \end{aligned}$$

(v) By (A4), $a + (-a) = 0 = (-a) + a$, since $-(-a)$ is unique by (i), we have $-(-a) = a$ by (A4).

(vi) Note that

$$\begin{aligned} (-a)(-b) &\overset{\text{(iv)}}{=} [(-1) \cdot a][(-1) \cdot b] \\ &\overset{(M1),(M2)}{=} [(-1) \cdot (-1)](a \cdot b) \\ &\overset{\text{(iv)}}{=} [-(-1)](a \cdot b) \\ &\overset{\text{(v)}}{=} 1 \cdot (a \cdot b) \\ &\overset{(M3)}{=} a \cdot b \end{aligned}$$

(vii) Note that

$$\begin{gathered} a > b \\ 0 \overset{(A4)}{=} a + (-a) \overset{(O2)}{>} b + (-a) \overset{(A1)}{=} -a + b \\ -b \overset{(A3)}{=} 0 + (-b) \overset{(O2)}{>} (-a + b) + (-b) \overset{(A2),(A4)}{=} -a + 0 \overset{(A3)}{=} -a \end{gathered}$$

(viii) Fixed any $c < 0$, by (vii), $-c > 0$. Hence, $-ca > -cb$ by (O3), so $ca < cb$ by (vii) and (v).

(ix) By (O1), there are two cases:

(Case 1) Suppose $a > 0$, then $a^2 \overset{(O3)}{>} a \cdot 0 \overset{\text{(iii)}}{=} 0$.

(Case 2) Suppose $a < 0$, then $a^2 \overset{\text{(viii)}}{>} a \cdot 0 \overset{\text{(iii)}}{=} 0$.

(x) Suppose it were not true that $1 > 0$, By (O1) and (D2), we have $1 < 0$.

By (M3), (vi), (ix), we have $1 = 1 \cdot 1 = (-1)^2 > 0$, which contradict with $1 < 0$ by (O1).

Therefore, $1 > 0$.

(xii) Note that $2 := 1 + 1 \overset{(O2),\text{(x)}}{>} 1 + 0 \overset{(A3)}{=} 1$. Hence, $2 > 0$ by (O4).

So $1 \overset{(M4)}{=} \frac{1}{2} \cdot 2 \overset{(O3)}{>} \frac{1}{2} \cdot 1 \overset{(M3)}{=} \frac{1}{2}$.

Suppose it were not true that $\frac{1}{2} > 0$. By (O1), there are two cases:

(Case 1) Suppose $\frac{1}{2} = 0$, then $1 \overset{(A4)}{=} 2 \cdot \frac{1}{2} \overset{\text{(iii)}}{=} 0$, which contradict with (D2).

(Case 2) Suppose $\frac{1}{2} < 0$, then $1 \overset{(A4)}{=} 2 \cdot \frac{1}{2} \overset{(O3)}{<} 2 \cdot 0 \overset{\text{(iii)}}{=} 0$, which contradict with (x) and (O1).

Hence, $\frac{1}{2} > 0$.

(xii) Suppose it were true that $a \neq 0$, by (O1) and assumption, $a > 0$,

Then, $a \overset{(M3)}{=} a \cdot 1 \overset{\text{(xi)},(O3)}{>} a \cdot \frac{1}{2} \overset{\text{(xi)},(O3)}{>} 0$, which contradict with the assumption if $\varepsilon = a \cdot \frac{1}{2}$.

Hence, $a = 0$.

1.3 Bernoulli's Inequality

If $x > -1$, then $(1 + x)^n \geq 1 + nx$ for any $n \in \mathbb{N}$.

Proof

Use Induction on n, it is obvious when $n = 1$.

Suppose the inequality holds for some $n = k \in \mathbb{N}$, i.e. $(1 + x)^k \geq 1 + kx$. Then

$$\begin{aligned}
(1 + x)^{k+1} &= (1 + x)(1 + x)^k \\
&\geq (1 + x)(1 + kx) && \text{By Induction Hypothesis} \\
&= 1 + kx + x + kx^2 \\
&\geq 1 + (k + 1)x && \text{since } x^2 \geq 0,
\end{aligned}$$

the statement is true when $n = k + 1$,

by principal of M.I., $(1 + x)^n \geq 1 + nx \ \forall\, n \in \mathbb{N}$.

Remark

With similar skill, we have if $x > -1$, then $(1 + x)^n \geq 1 + nx + \frac{1}{2}n(n - 1)x^2$ for any $n \in \mathbb{N}$ with $n \geq 2$.

1.4 Bounded Above and Below, Sup and Inf, Max and Min

1.4.1 Definition

Let $\varnothing \neq S \subset \mathbb{R}$. Then

(i) S is said to be bounded above (below resp.) if $\exists\, u \in \mathbb{R}$, s.t. $s \leq u \;\forall\, s \in S$ ($s \geq u \;\forall\, s \in S$ resp.).

In this case, u is called an upper (lower resp.) bound of S.

Also, S is said to be bounded if S is both bounded above and below.

(ii) Suppose S bounded above, $u \in \mathbb{R}$ is said to be a supremum of S, or we denote u as Sup$S$ if

- **(a)** u is an upper bound of S,
- **(b)** if v is another upper bound of S, then $v \geq u$.

(iii) Suppose S bounded below, $l \in \mathbb{R}$ is said to be an infimum of S, or we denote l as Inf$S$ if

- **(a)** l is a lower bound of S,
- **(b)** if k is another lower bound of S, then $l \geq k$.

(iv) Suppose S bounded above (below resp.), $u \in \mathbb{R}$ is said to be maximum (minimum resp.) of S, or we denote u as Max$S$ (Min$S$ resp.) if

- **(a)** $u \in S$,
- **(b)** $u \geq s \;\forall\, s \in S$ ($s \geq u \;\forall\, s \in S$ resp.).

remark

- Max$S$, Min$S$ may not exist even if S is bounded. (see example below)
- Sup$S$, Inf$S$, Max$S$, Min$S$ is unique if they exist. (Why?)

1.4.2 Property (equivalent definition of Sup)

Let u be an upper bound of $\varnothing \neq S \subset \mathbb{R}$.

Then $u =$Sup$S$ if and only if $\forall\, \varepsilon > 0,\ \exists\, s_0 \in S$, s.t. $s_0 > u - \varepsilon$.

Idea

A number is NOT an upper bound of S if it (strictly) less than u.

Proof

($\Longleftarrow$) Fixed any v be an upper bound of S. Suppose it were true that $v < u$.

Take $\varepsilon = u - v > 0$, by assumption, $\exists\, s_0 \in S$, s.t. $s_0 > u - \varepsilon = v$.

So v is NOT an upper bound, contradiction arise. Hence, $v \leq u$, so $u =$Sup$S$.

($\Longrightarrow$) Fixed any $\varepsilon > 0$, note that $u - \varepsilon < u$.

By def of Sup, $u - \varepsilon$ is NOT an upper bound of S.

Therefore, $\exists\, s_0 \in S$, s.t. $s_0 > u - \varepsilon$.

1.4.3 Corollary

If $M :=$Max$S$ exists in $\mathbb{R}$, then $M =$Sup$S$.

Proof

Note that $M > M - \varepsilon \;\forall\, \varepsilon > 0$ and $M \in S$, the result follow by last prop.

Remark

Similarly, we have the following property:

Let l be a lower bound of $\varnothing \neq S \subset \mathbb{R}$.

Then $l =$Inf$S$ if and only if $\forall\, \varepsilon > 0,\ \exists\, s_0 \in S$, s.t. $s_0 < l + \varepsilon$.

1.4.4 Example

Let $S = (-\infty, 1) := \{x \in \mathbb{R} : x < 1\}$, Show that S has no maximum and Sup$S = 1$.

Answer

Suppose S has the maximum M, then $M \in S$, i.e. $M < 1$. Let $M' = M + \frac{1}{2}(1 - M)$.

Since $1 - M > 0$ and $\frac{1}{2} > 0$, we have $M' > M$.

Since $1 - M > 0$ and $\frac{1}{2} < 1$, we have $M' < M + (1 - M) = 1$.

This means $M' \in S$ with $M' > M$, which contradict with M is the maximum of S.

So S has no maximum.

By def of S, we have $1 > s\ \forall\ s \in S$. Hence, S bounded above with an upper bound 1.

By Completeness Axiom of $\mathbb{R}$, Sup$S$ exists in $\mathbb{R}$. Fixed any $\varepsilon > 0$, define $s_0 = 1 - \frac{\varepsilon}{2}$.

Since $\varepsilon > 0$ and $\frac{1}{2} > 0$, so $s_0 = 1 - \frac{\varepsilon}{2} < 1$. Since $\varepsilon > 0$ and $\frac{1}{2} < 1$, so $s_0 = 1 - \frac{\varepsilon}{2} > 1 - \varepsilon$.

Therefore, $s_0 \in S$ with $s_0 > 1 - \varepsilon$, by prop 1.4.2 , Sup$S = 1$.

1.4.5 Property (Sup and subset)

Suppose $\emptyset \neq A \subset B \subset \mathbb{R}$, and $A,\ B$ bounded above, then Sup$A \leq$Sup$B$.

Proof

Let $u =$Sup$B$. Then $u \geq b\ \forall\ b \in B$.

In fact, since $A \subset B$, so $u \geq a\ \forall\ a \in A$, i.e. u is an upper bound of A.

By definition of Sup, Sup$B = u \geq$Sup$A$.

Challenging Question

Please define Sup$\emptyset$ and Inf$\emptyset$ and explain why.

1.4.6 Property (Sup and $+, \cdot$)

Let S, T be an bounded above subset of $\mathbb{R}$.

We define $a + S := \{a + s | s \in S\}$ and $aS := \{as | s \in S\}$ for any $a \in \mathbb{R}$.

Also, we define $S + T := \{s + t | s \in S,\ t \in T\}$.

Then

- **(i)** Sup$(a + S) = a+$Sup$S\ \forall\ a \in \mathbb{R}$,
- **(ii)** Sup$(aS) = a$Sup$S\ \forall\ a > 0$,
- **(iii)** Inf$(aS) = a$Sup$S\ \forall\ a < 0$. In particular, Inf$(-S) = -$Sup$S$,
- **(iv)** $S + T$ is bounded above with Sup $(S + T) =$ Sup$S$ + Sup$T$.

Proof

(i) Let $u =$Sup$S$. By def of Sup, $u > s\ \forall\ s \in S$.

Hence $a + u > as\ \forall\ s \in S$, i.e. $a + u > r\ \forall\ r \in a + S$.

Hence $a + S$ is bounded above with an upper bound $a + u$.

Using equivalent definition of Sup,

$\forall\ \varepsilon > 0,\ \exists\ s_0 \in S$, s.t. $s_0 > u - \varepsilon$.

Then, $\forall\, \varepsilon > 0,\ \exists\, s_0 \in S$, s.t. $a + s_0 > a + u - \varepsilon$.

Then, $\forall\, \varepsilon > 0,\ \exists\, r_0 \in a + S$, s.t. $r_0 > a + u - \varepsilon$.

Hence, $\text{Sup}(a + S) = a + u = a+\text{Sup}S$.

(ii) Let $u = \text{Sup}S$, $a > 0$. By def of Sup, $u > s\ \forall\, s \in S$.

Hence $au > as\ \forall\, s \in S$, i.e. $au > r\ \forall\, r \in aS$.

Hence aS is bounded above with an upper bound au.

Using equivalent definition of Sup,

$\forall\, \varepsilon > 0,\ \exists\, s_0 \in S$, s.t. $s_0 > u - \frac{\varepsilon}{a}$. Note that $\frac{\varepsilon}{a}>0$.

Then, $\forall\, \varepsilon > 0,\ \exists\, s_0 \in S$, s.t. $as_0 > au - \varepsilon$.

Then, $\forall\, \varepsilon > 0,\ \exists\, r_0 \in aS$, s.t. $r_0 > au - \varepsilon$.

Hence, $\text{Sup}(aS) = au = a\text{Sup}S$.

(iii) Let $u = \text{Sup}S$. By def of Sup, $u > s\ \forall\, s \in S$.

Hence $-u < -s\ \forall\, s \in S$, i.e. $-u < r\ \forall\, r \in -S$.

Hence $-S$ is bounded below with a lower bound $-u$.

Using equivalent definition of Sup and Inf,

$\forall\, \varepsilon > 0,\ \exists\, s_0 \in S$, s.t. $s_0 > u - \varepsilon$.

Then, $\forall\, \varepsilon > 0,\ \exists\, s_0 \in S$, s.t. $-s_0 < u + \varepsilon$.

Then, $\forall\, \varepsilon > 0,\ \exists\, r_0 \in -S$, s.t. $r_0 < u + \varepsilon$.

Hence, $\text{Inf}(-S) = -u = -\text{Sup}S$.

(iv) Let $u = \text{Sup}S$, $v = \text{Sup}T$.

By def of Sup, $u > s\ \forall\, s \in S$ and $v > t\ \forall\, t \in T$.

Then $u + v > s + t\ \forall\, s \in S,\ t \in T$, i.e. $u + v > r\ \forall r \in S + T$.

Hence $S + T$ is bounded above with an upper bound $u + v$.

Using equivalent definition of Sup,

$\forall\, \varepsilon > 0,\ \exists\, s_0 \in S,\ t_0 \in T$, s.t. $s_0 > u - \frac{\varepsilon}{2}$ and $t_0 > v - \frac{\varepsilon}{2}$.

Then, $\forall\, \varepsilon > 0,\ \exists\, s_0 \in S,\ t_0 \in T$, s.t. $s_0 + t_0 > u + v - \varepsilon$.

Then,$\forall\, \varepsilon > 0,\ \exists\, r_0 \in S + T$, s.t. $r_0 > u + v - \varepsilon$.

Hence, $\text{Sup}(S + T) = u + v = \text{Sup}S+\text{Sup}T$.

1.4.7 Definition (Bounded, Sup, Inf of Real-Valued Function)

Given $f : D \to \mathbb{R}$ be a real-valued function defined on D.

Then f is said to be bounded above (resp. below)

if the set $\{f(x) \in \mathbb{R} : x \in D\}$ is bounded above (resp. below).

An upper (resp. lower) bound of $\{f(x) \in \mathbb{R} : x \in D\}$

is also called an upper (resp. lower) bound of f on D.

f is said to be bounded if f is both bounded above and below.

If f is bounded above, We define Supremum of f on D by $\underset{x\in D}{\text{Sup}}\, f(x) = \text{Sup}\{f(x) \in \mathbb{R} : x \in D\}$.

If f is bounded below, We define Infimum of f on D by $\underset{x\in D}{\text{Inf}}\, f(x) = \text{Inf}\{f(x) \in \mathbb{R} : x \in D\}$.

1.4.8 Property

Given $f,\ g : D \to \mathbb{R}$ be a real-valued functions defined on D.

Note that $f + g$ is a real-valued functions defined on D

such that $(f+g)(x) = f(x) + g(x)\ \forall\ x \in D$. Then

(i) If $f(x) \le g(x)\ \forall\ x \in D$, Then $\operatorname{Sup}_{x\in D} f(x) \le \operatorname{Sup}_{x\in D} g(x)$.

(ii) $\operatorname{Sup}_{x\in D} (f+g)(x) \le \operatorname{Sup}_{x\in D} f(x) + \operatorname{Sup}_{x\in D} g(x)$.

Proof

(i) Let $G = \operatorname{Sup}_{x\in D} g(x)$.

Then by def of Sup, $G \ge g(x) \ge f(x)\ \forall\ x \in D$.

Then G is an upper bound of f on D.

By def of Sup, $\operatorname{Sup}_{x\in D} g(x) = G \ge \operatorname{Sup}_{x\in D} f(x)$.

(ii) Let $F = \operatorname{Sup}_{x\in D} f(x)$, $G = \operatorname{Sup}_{x\in D} g(x)$.

Then by def of Sup, $F \ge f(x)$ and $G \ge g(x)\ \forall\ x \in D$.

Hence $F + G \ge f(x) + g(x) = (f+g)(x)\ \forall\ x \in D$.

Then $F + G$ is an upper bound of $f + g$ on D.

By def of Sup, $\operatorname{Sup}_{x\in D} f(x) + \operatorname{Sup}_{x\in D} g(x) = F + G \ge \operatorname{Sup}_{x\in D} (f+g)(x)$.

Remark

The following statements are false, think about the counter example.

(i) If $f(x) \le g(x)\ \forall\ x \in D$, Then $\operatorname{Sup}_{x\in D} f(x) \le \operatorname{Inf}_{x\in D} g(x)$.

(ii) $\operatorname{Sup}_{x\in D} (f+g)(x) = \operatorname{Sup}_{x\in D} f(x) + \operatorname{Sup}_{x\in D} g(x)$.

1.5 Archimedean Property

1.5.1 Main Statement

$\forall\ x \in \mathbb{R},\ \exists\ n_x \in \mathbb{N}$, s.t. $x \le n_x$.

Equivalently, $\mathbb{N}$ is NOT bounded above.

Proof

Suppose it were true that $\mathbb{N}$ is bounded above.

By Completeness Axiom of $\mathbb{R}$, $u := \operatorname{Sup} \mathbb{N}$ exists.

By equivalent definition of Sup, $\exists\ m \in \mathbb{N}$, s.t. $m > u - 1$, i.e. $m + 1 > u$.

By def of $\mathbb{N}$, $m + 1 \in \mathbb{N}$, but $m + 1 > u$,

which is a contradiction. Hence, $\mathbb{N}$ is NOT bounded above.

1.5.2 Corollary

$$\text{Inf}\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0.$$

Equivalently, $\forall\, \varepsilon > 0,\ \exists\, n \in \mathbb{N}$, s.t. $0 < \frac{1}{n} < \varepsilon$.

Remark

This Corollary is sometimes referred to as the Archimedean Property.

Proof

Note that $\frac{1}{n} > 0\ \forall\, n \in \mathbb{N}$, so $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ bounded below with a lower bound 0.

By Completeness Axiom of $\mathbb{R}$, $w := \text{Inf}\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ exist in $\mathbb{R}$.

By def of Inf, $w \geq 0$.

$\forall\, \varepsilon > 0$, note that $\frac{1}{\varepsilon} > 0$, by Archimedean Property,

$\exists\, n \in \mathbb{N}$, s.t. $0 < \frac{1}{\varepsilon} < n$, i.e. $0 < \frac{1}{n} < \varepsilon$.

By def of Inf, $0 \leq w \leq \frac{1}{n} < \varepsilon$, this is true $\forall\, \varepsilon > 0$.

By Prop 1.2(xii), $\text{Inf}\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = w = 0$.

1.5.3 Example

Let $S = \left\{\frac{n}{2^n} : n \in \mathbb{N}\right\}$. Find Sup S, Inf S (If exist).

Answer

Note that $\frac{n+1}{2^{n+1}} \leq \frac{n+n}{2^{n+1}} = \frac{n}{2^n}$ true $\forall\, n \in \mathbb{N}$, so $\frac{n+1}{2^{n+1}} \leq \frac{n}{2^n} \leq ... \leq \frac{1}{2} \in S\ \forall\, n \in \mathbb{N}$.

Hence Max $S = \frac{1}{2}$, and so Sup $S = \frac{1}{2}$.

Note that $\frac{n}{2^n} > 0\ \forall\, n \in \mathbb{N}$. Then S is bounded below with lower bound 0.

By Completeness Axiom of $\mathbb{R}$, $w = \text{Inf } S$ exists in $\mathbb{R}$, and $w \geq 0$.

Fixed any $\varepsilon > 0$, by Archimedean Property, $\exists\, n' \in \mathbb{N}$, s.t. $\frac{1}{n'} < \frac{\varepsilon}{2}$, i.e. $\frac{2}{n'} < \varepsilon$. Then

$$0 \leq w \leq \frac{n'}{2^{n'}} = \frac{n'}{(1+1)^{n'}} \underset{Inequality}{\overset{Bernoulli's}{\leq}} \frac{n'}{1 + n' + \frac{1}{2}n'(n'-1)} = \frac{2}{\frac{2}{n'} + 2 + (n'-1)} = \frac{2}{n' + 1 + \frac{2}{n'}}$$
$$\leq \frac{2}{n'} < \varepsilon.$$

By Prop 1.2(xii), Inf $S = w = 0$.

1.6 Interval

1.6.1 Characterization of Interval

Let $\varnothing \neq S \subset \mathbb{R}$.

S is an interval if and only if $\forall\, x, y \in S$ with $x < y$, we have $[x, y] \subset S$.

1.6.2 Property (Union of Interval)

Let $\{I_n\}_{n=1}^{\infty}$ be sequence of interval.

If $\bigcap_{n=1}^{\infty} I_n := \{x \in \mathbb{R} : x \in I_n \ \forall\, n \in \mathbb{N}\}$ is non-empty,

then $\bigcup_{n=1}^{\infty} I_n := \{x \in \mathbb{R} : x \in I_n$ for some $n \in \mathbb{N}\}$ is an interval.

Proof

Let $z \in \bigcap_{n=1}^{\infty} I_n$. Pick any $x, y \in \bigcup_{n=1}^{\infty} I_n$ with $x < y$, we want to show $[x, y] \subset \bigcup_{n=1}^{\infty} I_n$.

By def of union, $\exists\, n_x, n_y$, s.t. $x \in I_{n_x}$ and $y \in I_{n_y}$.

By def of intersection, $z \in I_{n_x}$ and $z \in I_{n_y}$.

(Case 1) Suppose $x \leq z < y$.
By characterization of interval, $[x, z] \subset I_{n_x}$ and $[z, y] \subset I_{n_y}$.
Hence, $[x, y] = [x, z] \cup [z, y] \subset \bigcup_{n=1}^{\infty} I_n$.

(Case 2) Suppose $z < x < y$.
By characterization of interval, $[z, y] \subset I_{n_y}$.
Hence, $[x, y] \subset [z, y] \subset I_{n_y} \subset \bigcup_{n=1}^{\infty} I_n$.

(Case 3) Suppose $x < y \leq z$.
it is similarly with Case 2.

In any case, $[x, y] \subset \bigcup_{n=1}^{\infty} I_n$. By characterization of interval, $\bigcup_{n=1}^{\infty} I_n$ is an interval.

1.6.3 Nested Interval Theorem

Let $I_n := [a_n, b_n]$ be nested sequence (i.e. $I_{n+1} \subset I_n \ \forall\, n \in \mathbb{N}$) of CLOSED, BOUNDED intervals.

Then $\exists\, \xi \in \mathbb{R}$, s.t. $\xi \in I_n \ \forall\, n \in \mathbb{N}$. That is, $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

Furthermore, if the length of the intervals $b_n - a_n$ satisfy $\operatorname{Inf}\{b_n - a_n : n \in \mathbb{N}\} = 0$,

Then $\bigcap_{n=1}^{\infty} I_n$ is a singleton. That is, $\exists!\, \xi \in \mathbb{R}$, s.t. $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

1.6.4 Counter Example If Dropping Closed or Bounded Assumption

(Example 1) Let $I_n = \left(0, \frac{1}{n}\right) \ \forall\, n \in \mathbb{N}$. Note that $I_{n+1} \subset I_n \ \forall\, n \in \mathbb{N}$.

Hence, I_n is nested sequence of (bounded but not closed) intervals.

Suppose it were true that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$. Let $\xi \in \bigcap_{n=1}^{\infty} I_n$.

By def of I_n, $\xi > 0$. But by Archimedean Property, $\exists\, N \in \mathbb{N}$, s.t. $0 < \frac{1}{N} < \xi$.

It is a contradiction since $\xi \notin I_N$. Therefore, $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

(Example 2) Let $I_n = [n, +\infty)\ \forall\, n \in \mathbb{N}$. Note that $I_{n+1} \subset I_n\ \forall\, n \in \mathbb{N}$.

Hence, I_n is nested sequence of (closed but not bounded) intervals.

Suppose it were true that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$. Let $\xi \in \bigcap_{n=1}^{\infty} I_n$.

Note that $\xi \in \mathbb{R}$. But by Archimedean Property, $\exists\ N \in \mathbb{N}$, s.t. $\xi \leq N$.

It is a contradiction since $\xi \notin I_N$. Therefore, $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

2 Sequences

2.1 Definition and Basic Property

2.1.1 Definition (Sequence)

A sequence in $\mathbb{R}$ is a function $a : \mathbb{N} \to \mathbb{R}$.

We usually write $a(n)$ as a_n. Also, we write the sequence a as

$$\{a_n\},\ (a_n),\ \{a_n\}_{n=1}^{\infty} \text{ or } (a_n)_{n=1}^{\infty}$$

2.1.2 Definition (Limit of Sequence)

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. We say x_n converge to $L \in \mathbb{R}$ if

$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$, s.t. $\forall\ n \in \mathbb{N}$ with $n \geq N$, we have $\left|x_n - L\right| < \varepsilon$.

In this case, we say L is a limit of x_n and x_n is a convergent sequence.

If x_n has no limit in $\mathbb{R}$, then we say x_n is a divergent sequence.

Remark

(i) When the question need you to prove L is the limit of sequence,

you CANNOT determine the value of ε, you only know ε is arbitrary (small) positive number,

and then find a (large) N (depends on ε) satisfy the result.

(ii) When the question give you the result that $L = \lim_n x_n$,

you can take any positive number of ε,

could be 1, $\frac{|x|}{2}$ (for some $x \neq 0$), or just write $\varepsilon > 0$, depends on what is the conclusion.

then the assumption will give you a (large) N (you don't know what this N is),

such that $\left|x_n - L\right| \leq \varepsilon\ \forall\ n \geq N$, and then using this fact to prove the result.

(iii) x_n is divergent if $\forall\ L \in \mathbb{R},\ \exists\ \varepsilon_0 > 0$, s.t. $\forall\ N \in \mathbb{N},\ \exists\ n' \geq N$, s.t. $\left|x_{n'} - L\right| \geq \varepsilon_0$.

2.1.3 Property (Uniqueness of Limit)

Limit of a convergent sequence in $\mathbb{R}$ is unique.

Therefore, if $L \in \mathbb{R}$ is the limit of $\{x_n\}$, we will write in this notation:

$$\lim_n x_n = L \quad \text{OR} \quad x_n \to L \ \text{ as } n \to \infty.$$

Proof

Let $L, L' \in \mathbb{R}$ be limits of a convergent sequence x_n. Pick any $\varepsilon > 0$,

$\exists\ N \in \mathbb{N}$, s.t. $\forall\ n \geq N$, we have $\left|x_n - L\right| < \frac{\varepsilon}{2}$,

$\exists\ N' \in \mathbb{N}$, s.t. $\forall\ n \geq N'$, we have $\left|x_n - L'\right| < \frac{\varepsilon}{2}$.

Take $M = \text{Max}\ \{N, N'\}$,

Then $\left|L - L'\right| \leq \left|L - x_M\right| + \left|x_M - L'\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

This true for any $\varepsilon > 0$, so $\left|L - L'\right| = 0$. Hence, $L = L'$.

2.1.4 Example

Determine the following sequences are convergent / divergent.

If convergent, guess the limit and prove it by $\varepsilon - N$ definition. If divergent, give a reason.

(a) $a_n = \frac{1}{n}$,

(b) $a_n = (-1)^n$,

(c) $a_n = \frac{5n+2}{n+1}$,

(d) $a_n = r^n$ given that $0 < r < 1$.

Answer

(a) Guess a_n converge to 0.

Fixed any $\varepsilon > 0$, by A.P., $\exists\ N \in \mathbb{N}$, s.t. $0 < \frac{1}{N} < \varepsilon$.

Note that $\forall\ n \geq N$, we have $0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$,

that means $\forall\ n \geq N$, we have $\left|a_n - 0\right| = \frac{1}{n} < \varepsilon$.

Hence, $\{a_n\}$ convergent with $\lim_n a_n = 0$.

(b) Guess a_n divergent.

Fixed any $L \in \mathbb{R}$, take $\varepsilon_0 = \frac{1}{2}\text{ Max}\left\{\ \left|L-1\right|,\ \left|L+1\right|\ \right\} > 0$, fixed any $N \in \mathbb{N}$,

(Case 1) Suppose $\varepsilon_0 = \frac{1}{2}\left|L-1\right| > 0$.

Take $n' = 2N \geq N$, then $\left|a_{n'} - L\right| = \left|1 - L\right| = \left|L - 1\right| \geq \varepsilon_0$.

(Case 2) Suppose $\varepsilon_0 = \frac{1}{2}\left|L+1\right| > 0$.

Take $n' = 2N + 1 \geq N$, then $\left|a_{n'} - L\right| = \left|-1 - L\right| = \left|L + 1\right| \geq \varepsilon_0$.

In any case, we can find $n' \geq N$ s.t. $\left|a_{n'} - L\right| \geq \varepsilon_0$, hence, $\{a_n\}$ divergent.

(c) Guess a_n converge to 5.

Fixed any $\varepsilon > 0$, by A.P., $\exists\ N \in \mathbb{N}$, s.t. $0 < \frac{1}{N} < \frac{\varepsilon}{3}$.

Note that $\forall\ n \geq N$, we have $0 < \frac{3}{n} \leq \frac{3}{N} < \varepsilon$,

that means $\forall\ n \geq N$, we have $\left|a_n - 5\right| = \left|\frac{-3}{n+1}\right| < \frac{3}{n} < \varepsilon$.

Hence, $\{a_n\}$ convergent with $\lim_n a_n = 5$.

(d) Guess a_n converge to 0. [We want to use Bernoulli's Inequality.]

Let $q = \frac{1}{r} - 1 > 0$, then $r = \frac{1}{q+1}$.

Fixed any $\varepsilon > 0$, by A.P., $\exists\ N \in \mathbb{N}$, s.t. $0 < \frac{1}{N} < q\varepsilon$.

Note that $\forall\ n \geq N$, we have $0 < \frac{1}{nq} \leq \frac{1}{Nq} < \varepsilon$,

that means $\forall\ n \geq N$, we have $\left|a_n - 0\right| = r^n = \frac{1}{(q+1)^n} \underset{Inequality}{\overset{Bernoulli's}{\leq}} \frac{1}{1+nq} \leq \frac{1}{nq} < \varepsilon$.

Hence, $\{a_n\}$ convergent with $\lim_n a_n = 0$.

2.1.5 Definition (Bounded)

A sequence x_n is said to be bounded if $\exists\ M > 0$, s.t. $|x_n| < M\ \forall\ n \in \mathbb{N}$.

2.1.6 Property

Convergent sequence must be bounded.

Proof

Let $\{x_n\}$ be convergent sequence with limit $x \in \mathbb{R}$.

Take $\varepsilon = 1,\ \exists\ N \in \mathbb{N}$, s.t. $|x_n - x| < \varepsilon = 1\ \forall\ n \geq N$.

i.e. $x - 1 < x_n < x + 1\ \forall\ n \geq N$.

i.e. $|x_n| <$ Max $\{|x-1|,\ |x+1|\}\ \ \forall\ n \geq N$. (*Remark*: it is necessary since $x + 1$ can be negative.)

Hence, $|x_n| <$ Max $\{|x_1|, |x_2|, ..., |x_{N-1}|, |x-1|, |x+1|\}\ \ \forall\ n \in \mathbb{N}$

(*Remark*: This Max exist in $\mathbb{R}$ since the set is finite.)

Hence, $\{x_n\}$ is bounded.

Remark

The converse is not true, the counter example is 2.1.4(b),

the sequence is bounded but not convergent.

2.1.7 Property

Fixed some $m \in \mathbb{N}$.

$\{x_n\}_{n=1}^{\infty}$ is a convergent sequence if and only if $\{x_{n+m}\}_{n=1}^{\infty}$ is also a convergent sequence.

In this case, $\lim_n x_n = \lim_n x_{n+m}$.

Idea

The limit/convergence of a sequence describe the mass behaviour of the terms for all n large,

it will NOT be affected by finitely many terms.

Proof

($\Longrightarrow$) Suppose x_n converge to $x \in \mathbb{R}$.

Then fixed any $\varepsilon > 0,\ \exists\ N \in \mathbb{N}$, s.t. $\forall\ n \geq N$, we have $|x_n - x| < \varepsilon$.

In particular, we have $|x_{n+m} - x| < \varepsilon\ \forall\ n + m \geq N$.

That is we have $|x_{n+m} - x| < \varepsilon\ \forall\ n \geq N$. (since $m \geq 1$.)

Hence, we have x_{n+m} converge to x.

($\Longleftarrow$) Suppose x_{n+m} converge to $x \in \mathbb{R}$.

Then fixed any $\varepsilon > 0,\ \exists\ N \in \mathbb{N}$, s.t. $\forall\ n \geq N$, we have $|x_{n+m} - x| < \varepsilon$.

Let $N' = N + m \in \mathbb{N}$, then we have $|x_n - x| < \varepsilon\ \forall\ n \geq N'$.

Hence, we have x_n converge to x.

2.1.8 Property

Let $\{x_n\}$ be a convergent sequence with $\lim_n x_n = x$.

If $\alpha < x < \beta$ for some $\alpha, \beta \in \mathbb{R}$, show that $\exists\ N \in \mathbb{N}$ s.t. $\alpha < x_n < \beta\ \forall\ n \geq N$.

Proof

Take $\varepsilon_0 = \text{Min}\{\beta - x, x - \alpha\} > 0$, by x_n converge to x,

$\exists\ N \in \mathbb{N}$, s.t. $|x_n - x| < \varepsilon_0\ \forall\ n \geq N$,

that is $x - \varepsilon_0 < x_n < x + \varepsilon_0\ \forall\ n \geq N$.

Note that $\varepsilon \leq \beta - x$ and $\varepsilon \leq x - \alpha$ by definition of Min.

Hence, $\alpha = x - (x - \alpha) \leq x - \varepsilon_0 < x_n < x + \varepsilon_0 \leq x + (\beta - x) = \beta\ \forall\ n \geq N$.

2.2 Monotone Convergent Theorem

2.2.1 Definition

- A sequence $\{x_n\}$ is said to be increasing if $x_n \leq x_{n+1}\ \forall\ n \in \mathbb{N}$.
- A sequence $\{x_n\}$ is said to be decreasing if $x_n \geq x_{n+1}\ \forall\ n \in \mathbb{N}$.
- A sequence is said to be monotone if it is increasing or decreasing.

2.2.2 Main Statement of Theorem

- An increasing sequence $\{x_n\}$ is convergent if and only if it is bounded above. In this case,

$$\lim_n x_n = \text{ Sup } \{x_n : n \in \mathbb{N}\}$$

- An decreasing sequence $\{x_n\}$ is convergent if and only if it is bounded below. In this case,

$$\lim_n x_n = \text{ Inf } \{x_n : n \in \mathbb{N}\}$$

Remark

The theorem is still true if the tail of the sequence is monotone.

2.2.3 Example

Let $x_1 = 8, x_{n+1} = \frac{1}{2}x_n + 2\ \forall\ n \in \mathbb{N}$. Show $\{x_n\}$ convergent and find the limit.

Answer

Use induction on *n* to show the sequence is decreasing and bounded below by 0.

Note $0 < x_2 = 6 \leq 8 = x_1$. Now assume $0 < x_k \leq x_{k-1}$ for some $k \in \mathbb{N}$.

Then $x_{k+1} = \frac{1}{2}x_k + 2 \leq \frac{1}{2}x_{k-1} + 2 = x_k$ and $x_{k+1} = \frac{1}{2}x_k + 2 > 0 + 2 > 0$.

Then $\{x_n\}$ is a bounded below decreasing sequence and

hence convergent by Monotone Convergent Theorem.

Let $x = \lim_n x_n$, then we have

$$\begin{aligned} \lim_n x_{n+1} &= \frac{1}{2} \lim_n x_n + 2 \\ x &= \frac{1}{2}x + 2 \\ x &= 4. \end{aligned}$$

2.3 Bolzano-Weierstrass Theorem

2.3.1 Definition

Let $\left\{x_n\right\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and

$\left\{n_k\right\}_{k=1}^{\infty}$ be a STRICTLY increasing sequence in $\mathbb{N}$. (i.e $n_1 < n_2 < ...$ and $n_k \in \mathbb{N}\ \forall\ k \in \mathbb{N}$)

The sequence $\left\{x_{n_k}\right\}_{k=1}^{\infty}$ is called a subsequence of $\left\{x_n\right\}$.

2.3.2 Example

Let $x_n = \dfrac{1}{2n+3}$, $n_k = k^2$, the subsequence can be expression by this table:

k	1	2	3	k
n_k	1	4	9	k^2
x_{n_k}	$\dfrac{1}{2+3} = \dfrac{1}{5}$	$\dfrac{1}{8+3} = \dfrac{1}{11}$	$\dfrac{1}{18+3} = \dfrac{1}{21}$	$\dfrac{1}{2k^2+3}$

2.3.3 Property

Let $\left\{x_{n_k}\right\}$ be subsquence of $\left\{x_n\right\}$ in $\mathbb{R}$. Then

(i) $n_k \geq k\ \forall\ k \in \mathbb{N}$.

(ii) if $\left\{x_n\right\}$ converge, then $\left\{x_{n_k}\right\}$ converge to same limit.

Proof

(i) Use Induction on k, it is true when $k = 1$ since Min $\mathbb{N} = 1$.

Assume $n_l \geq l$ for some $l \in \mathbb{N}$, then $n_{l+1} > n_l \geq l$, so $n_{l+1} \geq l + 1$. (Why?)

Hence, $n_k \geq k\ \forall\ k \in \mathbb{N}$.

(ii) Suppose $\lim_n x_n = x \in \mathbb{R}$. Fixed any $\varepsilon > 0$, we have some $N \in \mathbb{N}$, s.t. $\left|x_n - x\right| < \varepsilon\ \forall\ n \geq N$.

In particular, by (i), if $k \geq N$, $n_k \geq N$, so $\left|x_{n_k} - x\right| < \varepsilon\ \forall\ k \geq N$. That is, $\lim_k x_{n_k} = x$.

2.3.4 Corollary

If the sequence $\left\{x_n\right\}$

(i) has a divergent subsequence, OR

(ii) has two convergent subsequence $\left\{x_{n_i}\right\}$, and $\left\{x_{n_j}\right\}$ with $\lim_i x_{n_i} \neq \lim_j x_{n_j}$,

then $\left\{x_n\right\}$ is divergent.

2.3.5 Claim

Every sequence in $\mathbb{R}$ has a monotone subsequence.

Proof

Let $\left\{x_n\right\}$ be a sequence in $\mathbb{R}$. We define x_m is a "peak" if $x_m \geq x_n\ \forall\ m \leq n$.

(Case 1) Suppose $\left\{x_n\right\}$ has infinitely many "peaks".
Then list the "peaks" $x_{m_1}, x_{m_2}, ..., x_{m_k}, ...$ with $m_1 < m_2 < ... < m_k < ...$.
By definition of "peak", we have $x_{m_1} \geq x_{m_2} \geq ... \geq x_{m_k} \geq ...$,
hence $\left\{x_{m_k}\right\}$ is a decreasing subsequence.

(Case 2) Suppose $\{x_n\}$ has finitely many "peaks".

Then list ALL "peaks" $x_{m_1}, x_{m_2}, ..., x_{m_N}$ with $m_1 < m_2 < ... < m_N$.

That means x_n is NOT a "peak" if $n > N$.

Take $n_1 = N + 1 > N$, since x_{n_1} is not a "peak", then $\exists\, n_2 > n_1$, s.t. $x_{n_2} > x_{n_1}$.

Since $n_2 > n_1 > N$, then x_{n_2} is not a "peak", then $\exists\, n_3 > n_2 > n_1$, s.t. $x_{n_3} > x_{n_2} > x_{n_1}$.

Repeat the process, we have $N < n_1 < n_2 < ... < n_k < ...$

such that $x_{n_1} < x_{n_2} < ... < x_{n_k} < ...$

that means $\left\{x_{n_k}\right\}$ is a (strictly) incresing subsequence.

2.3.6 Bolzano-Weierstrass Theorem

Every bounded sequence has convergent subsequence.

Proof (from Monotone Convergent Theorem)

Let $\{x_n\}$ be bounded sequence. By the claim, there are a monotone subsequence $\left\{x_{n_k}\right\}$.

Since $\{x_n\}$ bounded, so $\left\{x_{n_k}\right\}$ bounded. (Why?)

By Monotone Convergent Theorem, $\left\{x_{n_k}\right\}$ converge.

2.4 Cauchy Convergent Theorem

2.4.1 Definition

A sequence in $\mathbb{R}$ is said to be Cauchy if

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\ \text{s.t.}\ \forall\, n, m \geq N,\ \text{we have}\ |x_n - x_m| < \varepsilon.$$

2.4.2 Main Statement of Theorem

A sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy.

2.5 Properly Divergent and Series

2.5.1 Definition

(i) A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be tends to $+\infty$, denoted as $\lim_n x_n = +\infty$,

if $\forall\, M > 0,\ \exists\, N \in \mathbb{N}$, s.t. $\forall\, n \geq N$, we have $x_n > M$.

(ii) A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be tends to $-\infty$, denoted as $\lim_n x_n = -\infty$,

if $\forall\, M > 0,\ \exists\, N \in \mathbb{N}$, s.t. $\forall\, n \geq N$, we have $x_n < -M$.

(iii) In this two cases, the sequence is called properly divergent.

2.5.2 Example involving summation

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Define $\{S_n\}$ by

$$S_n = \frac{1}{n}\left(x_1 + x_2 + ... + x_n\right) = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

that is the mean of first n terms.

(a) If $\lim_n x_n = x \in \mathbb{R}$, show that $\lim_n S_n = x$.

(b) If $\lim_n x_n = +\infty$, what can you say about $\lim_n S_n$? Provide the reason.

(c) Is that true that $\{x_n\}$ is convergent given that $\{S_n\}$ is convergent?

Answer

(a) Fixed any $\varepsilon > 0$,

by $\lim_n x_n = x$, $\exists\, N_1 \in \mathbb{N}$, s.t. $|x_n - x| < \frac{\varepsilon}{2}$ $\forall\, n \geq N_1$.

Now $K := \sum_{i=1}^{N_1} |x_i - x| + 1$ is a fixed constant, by A.P., $\exists\, N_2 \in \mathbb{N}$, s.t. $\frac{1}{N_2} < \frac{\varepsilon}{2K}$.

Take $N = \text{Max}\,\{N_1, N_2\}$. If $n \geq N$, we have

$$\begin{aligned}
|S_n - x| &= \frac{1}{n}\left|\sum_{i=1}^{n} x_i - nx\right| = \frac{1}{n}\left|\sum_{i=1}^{n} (x_i - x)\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}|x_i - x| \\
&= \frac{1}{n}\sum_{i=1}^{N_1}|x_i - x| + \frac{1}{n}\sum_{i=N_1+1}^{n}|x_i - x| \\
&< \frac{1}{N_2}K + \frac{1}{n}\sum_{i=N_1+1}^{n}\frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{n - N_1}{n}\frac{\varepsilon}{2} \\
&\leq \varepsilon.
\end{aligned}$$

Hence, we have $\lim_n S_n = x$.

(b) Guess $\lim_n S_n = +\infty$. Fixed any $M > 0$,

by $\lim_n x_n = +\infty$, $\exists\, N_1 \in \mathbb{N}$, s.t. $x_n > 3M$ $\forall\, n \geq N_1$.

Now $K := \sum_{i=1}^{N_1} |x_i|$ is a fixed constant, by A.P., $\exists\, N_2 \in \mathbb{N}$, s.t. $\frac{K}{M} < N_2$.

Note $x_i \geq -|x_i|$ $\forall\, i = 1, 2, ..., N_1 - 1$, so $\frac{1}{n}\sum_{i=1}^{N_1} x_i \geq -\frac{1}{n}\sum_{i=1}^{N_1}|x_i| \geq -\frac{K}{N_2} \geq -M$ $\forall\, n \geq N_2$.

Take $N = \text{Max}\,\{3N_1, N_2\}$. If $n \geq N$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i &= \frac{1}{n}\sum_{i=1}^{N_1} x_i + \frac{1}{n}\sum_{i=N_1+1}^{n} x_i \\
&> -M + \frac{n - N_1}{n}(3M) \\
&= -M + \left(1 - \frac{N_1}{n}\right)(3M) \\
&\geq -M\left(1 - \frac{N_1}{3N_1}\right)(3M) \\
&= -M + \frac{2}{3}\cdot 3M \\
&= M
\end{aligned}$$

Hence, we have $\lim_n S_n = +\infty$.

(c) NO. Consider the counter example $x_n = (-1)^n$,

Note $\{x_n\}$ is NOT a convergent sequence but $S_n = \begin{cases} \frac{-1}{n} & \text{, if } n \text{ is odd} \\ 0 & \text{, if } n \text{ is even} \end{cases}$ converge to 0.

Limit Superior and Limit Inferior

2.5.3 Definition

Let $\{x_n\}$ be a BOUNDED sequence in $\mathbb{R}$. We define

- $\limsup_n x_n = \lim_n \sup_{k\geq n} x_k$,
- $\liminf_n x_n = \lim_n \inf_{k\geq n} x_k$.

2.5.4 Equivalent Definition

Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Then $\limsup_n x_n = x$ is equivalent to

(i) $x = \limsup_n x_n = \lim_n \sup_{k\geq n} x_k = \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k$, OR

(ii) $\forall \varepsilon > 0$, $x + \varepsilon < x_n$ for ONLY finitely many $n \in \mathbb{N}$
but $x - \varepsilon < x_n$ for INFINTELY many $n \in \mathbb{N}$.

2.5.5 Property

Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Then

$\{x_n\}$ is convergent if and only if $\limsup_n x_n = \liminf_n x_n$.

In this case, we have $\limsup_n x_n = \lim_n x_n = \liminf_n x_n$.

Proof

($\Longrightarrow$) Suppose $\lim_n x_n = x \in \mathbb{R}$. Fixed any $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$, s.t. $|x_n - x| < \frac{\varepsilon}{2}\ \forall\ n \geq N$.

That is, $x - \frac{\varepsilon}{2} < x_n < x + \frac{\varepsilon}{2}\ \forall\ n \geq N$. Therefore, for any $n \geq N$, we have

$$x - \frac{\varepsilon}{2} < \sup_{k\geq n} x_k \leq x + \frac{\varepsilon}{2} \quad \text{and} \quad x - \frac{\varepsilon}{2} \leq \inf_{k\geq n} x_k < x + \frac{\varepsilon}{2}.$$

Hence, we have $\left|\sup_{k\geq n} x_k - x\right| \leq \frac{\varepsilon}{2} < \varepsilon$ and $\left|\inf_{k\geq n} x_k - x\right| \leq \frac{\varepsilon}{2} < \varepsilon\ \forall\ n \geq N$.

Hence, $\limsup_n x_n = x = \liminf_n x_n$.

($\Longleftarrow$) Suppose $\limsup_n x_n = \liminf_n x_n = x \in \mathbb{R}$. Fixed any $\varepsilon > 0$,

$\exists\ N_1 \in \mathbb{N}$, s.t. $\left|\sup_{k\geq n} x_k - x\right| < \varepsilon\ \forall n \geq N_1$, in particular, $\sup_{k\geq n} x_k < x + \varepsilon\ \forall\ n \geq N_1$.

$\exists\ N_2 \in \mathbb{N}$, s.t. $\left|\inf_{k\geq n} x_k - x\right| < \varepsilon\ \forall n \geq N_2$, in particular, $\inf_{k\geq n} x_k > x - \varepsilon\ \forall\ n \geq N_2$.

Hence, for any $n \geq N := \text{Max}\,\{N_1, N_2\}$, we have

$$x - \varepsilon < \inf_{k\geq N} x_k \leq x_n \leq \sup_{k\geq N} x_k < x + \varepsilon.$$

That is, we have $|x_n - x| \leq \varepsilon\ \forall\ n \geq N$.

Therefore, $\{x_n\}$ is convergent with $\lim_n x_n = x$.

2.5.6 Property

Let $\{x_n\}, \{y_n\}$ be bounded sequences in $\mathbb{R}$. Then

$$\limsup_n (x_n + y_n) \leq \limsup_n x_n + \limsup_n y_n.$$

Proof

Note for any $n \in \mathbb{N}$, $x_m + y_m \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k \ \forall\, m \geq n$,

Hence $\sup_{k \geq n} (x_k + y_k) \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k \ \forall\, n \in \mathbb{N}$. Therefore,

$$\limsup_n (x_k + y_k) \leq \lim_n \left(\sup_{k \geq n} x_k + \sup_{k \geq n} y_k \right) = \limsup_n x_n + \limsup_n y_n.$$

Remark

The inequality may be occur. Think about $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

3 Limit of Function

3.1 Basic Property

3.1.1 Definition (Neighborhood)

Let $c \in \mathbb{R}$, $\delta > 0$, we denote the $\delta-$neighborhood of c as

$$V_\delta := (c - \delta, c + \delta) = \left\{x \in \mathbb{R} : |x - c| < \delta\right\}.$$

3.1.2 Definition (Cluster Point)

Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a cluster point w.r.t. A if

$$\forall\, \varepsilon > 0,\ \exists\, x \in A \text{ with } x \neq c,\ \text{s.t. } |x - c| < \varepsilon \text{ (Or } x \in V_\varepsilon(c) \setminus \{c\}).$$

Remark

A cluster point $c \in \mathbb{R}$ w.r.t. A may NOT be in A. (Consider $A = \mathbb{R} \setminus \{0\},\ c = 0$)

A point $a \in A$ may NOT be a cluster point w.r.t A. (Consider $A = \{0\},\ a = 0$)

3.1.3 Definition (Limit of Function)

Let $\varnothing \neq A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $c \in \mathbb{R}$ be a cluster point w.r.t. A.

$L \in \mathbb{R}$ is said to be a limit of f at c if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \text{s.t. } \forall\, x \in (c - \delta, c + \delta) \text{ with } x \neq c,\ \text{we have } \left|f(x) - L\right| < \varepsilon.$$

By some property, we know the limit of f at c is unique if it exists,

hence we will denote the above case as

$$\lim_{x \to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c$$

3.1.4 Definition

Sometime, we will discuss different types of limit of function.

For example, we will discuss f tends to infinity or as x tends to infinity or the one-sided limit.

It will be difficult to remember all the cases. But the patterns of them are similar.

$\lim_{x \to c} f(x) = L$ if $\forall$ Statement A, $\exists$ Statement B, s.t. $\forall\, x \in \mathbb{R}$ with Statement C, we have Statement D.

Cases	Notation	Statement A	Statement B	Statement C	Statement D
Two-sided limit	$\lim_{x \to c} f(x) =??$	-	$\delta > 0$	$0 < \|x - c\| < \delta$	-
RHS one-sided limit	$\lim_{x \to c^+} f(x) =??$	-	$\delta > 0$	$0 < x - c < \delta$	-
LHS one-sided limit	$\lim_{x \to c^-} f(x) =??$	-	$\delta > 0$	$0 < c - x < \delta$	-
limit as $x \to +\infty$	$\lim_{x \to +\infty} f(x) =??$	-	$N > 0$	$x \geq N$	-
limit as $x \to -\infty$	$\lim_{x \to -\infty} f(x) =??$	-	$N > 0$	$x \leq -N$	-
limit tends to $L \in \mathbb{R}$	$\lim_{x \to ??} f(x) = L$	$\varepsilon > 0$	-	-	$\left\|f(x) - L\right\| < \varepsilon$
limit tends to $+\infty$	$\lim_{x \to ??} f(x) = +\infty$	$M > 0$	-	-	$f(x) > M$
limit tends to $-\infty$	$\lim_{x \to ??} f(x) = -\infty$	$M > 0$	-	-	$f(x) < -M$

Example

$\lim_{x \to 2^-} f(x) = +\infty$ means $\forall\, M > 0,\ \exists\, \delta > 0$, s.t. $\forall\, x \in \mathbb{R}$ with $0 < x - 2 < \delta$, we have $f(x) > M$.

3.1.5 Example

Guess the limit and proof by definition.

(i) $\lim_{x \to -1} \frac{x^2}{x+2}$ (Ans: 1)

(ii) $\lim_{x \to 2} \frac{x^3+3}{x-1}$ (Ans: 11)

(iii) $\lim_{x \to 1^-} \frac{x}{x-1}$ (Ans: $-\infty$)

(iv) $\lim_{x \to -\infty} \frac{x^2}{2x^2-1}$ (Ans: $\frac{1}{2}$)

Answer

(i) Fixed any $\varepsilon > 0$, take $\delta = \text{Min}\left\{\frac{1}{2}, \frac{\varepsilon}{8}\right\} > 0$, if $x \in \mathbb{R}$ with $0 < \left|x+1\right| < \delta$, we have

$$-1-\delta < x < -1+\delta$$
$$-\frac{3}{2} < x < -\frac{1}{2} < 0.$$

That is, $0 < \frac{1}{2} < x+2 < 2$ and hence $\frac{1}{2} < \frac{1}{x+2} < 2$, and also, $|x| < \frac{3}{2} < 2$.

If $x \in \mathbb{R}$ with $0 < \left|x+1\right| < \delta$, we have

$$\left|\frac{x^2}{x+2} - 1\right| = \left|\frac{x^2-x-2}{x+2}\right| = \left|x-1\right|\left|\frac{x-2}{x+2}\right| \leq 2\delta\left(|x|+2\right) \leq 8\delta < \varepsilon.$$

Hence, $\lim_{x \to -1} \frac{x^2}{x+2} = 1$.

(ii) Fixed any $\varepsilon > 0$, take $\delta = \text{Min}\left\{\frac{1}{2}, \frac{\varepsilon}{40}\right\} > 0$, if $x \in \mathbb{R}$ with $0 < \left|x-2\right| < \delta$, we have

$$2-\delta < x < 2+\delta$$
$$0 < \frac{3}{2} < x < \frac{5}{2}.$$

That is, $0 < \frac{1}{2} < x-1 < \frac{3}{2}$ and hence $\frac{2}{3} < \frac{1}{x-1} < 2$, and also, $|x|^2 + 2|x| + 7 < \frac{25}{4} + 5 + 7 < 20$.

If $x \in \mathbb{R}$ with $0 < \left|x-2\right| < \delta$, we have

$$\left|\frac{x^3+3}{x-1} - 11\right| = \left|\frac{x^3-11x+14}{x-1}\right| = \left|x=2\right|\left|\frac{x^2+2x-7}{x-1}\right| \leq 2\delta\left(|x|^2+2|x|+7\right) \leq 40\delta < \varepsilon.$$

Hence, $\lim_{x \to -1} \frac{x^3+3}{x-1} = 11$.

(iii) Fixed any $M > 0$, take $\delta = \frac{1}{M+1} > 0$, if $x \in \mathbb{R}$ with $0 < 1-x < \delta$, we have

$$-x < -1 - \frac{1}{M+1} = -\frac{M}{M+1}$$
$$x > \frac{M}{M+1}$$
$$Mx + x > M \qquad \text{Since } M+1 > 0$$
$$x > -M(x-1)$$
$$\frac{x}{x-1} < -M \qquad \text{Since } x-1 < 0$$

(iv) Fixed any $\varepsilon > 0$, by A.P., $\exists M \in \mathbb{N}$, s.t. $\dfrac{1}{M} < \varepsilon$, W.L.O.G, assume $M \geq 2$.

If $x < -M$, then $x^2 > M^2 > M$, and so

$$\left|\frac{x^2}{2x^2-1} - \frac{1}{2}\right| = \left|\frac{1}{2(2x^2-1)}\right| \leq \frac{1}{4M^2-2} \leq \frac{1}{M} \leq \varepsilon.$$

Hence, $\lim\limits_{x\to-\infty} \dfrac{x^2}{2x^2-1} = \dfrac{1}{2}$.

3.2 Sequential Criterion

3.2.1 Sequential Criterion for Limit of Function

Let $f : A \to \mathbb{R}$, $c \in \mathbb{R}$ is a cluster point of A. Let $L \in \mathbb{R}$. Then

$\lim\limits_{x\to c} f(x) = L$ if and only if

$\lim\limits_{n} f(a_n) = L$ for any sequence $\{a_n\}$ with $a_n \in A \setminus \{c\}$ $\forall\, n \in \mathbb{N}$ and $\lim\limits_{n} a_n = c$.

3.2.2 Sequential / Cauchy Criterion for Limit of Function

Let $f : A \to \mathbb{R}$, $c \in \mathbb{R}$ is a cluster point of A. Let $L \in \mathbb{R}$.

The following statements are equivalent:

(i) $\lim\limits_{x\to c} f(x)$ exists in $\mathbb{R}$.

(ii) *(Sequential Criterion)*
$\lim\limits_{n} f(x_n)$ exists for any sequence $\{x_n\}$ with $x_n \in A \setminus \{c\}$ $\forall\, n \in \mathbb{N}$ and $\lim\limits_{n} x_n = c$.
(the limits are NOT necessarily same for each sequence, but in fact they are same.)

(iii) *(Cauchy Criterion)*
$\forall\, \varepsilon > 0,\ \exists\, \delta > 0$, s.t. $\forall\, x, x' \in A$ with $0 < |x - c| < \delta$ and $0 < |x' - c| < \delta$,
we have $|f(x) - f(x')| < \varepsilon$.

Proof

(i) $\Longrightarrow$ (iii) Suppose $\lim\limits_{x\to c} f(x) = L \in \mathbb{R}$. Fixed any $\varepsilon > 0$,

we can find some $\delta > 0$, such that $|f(w) - L| < \dfrac{\varepsilon}{2}$ $\forall\, w \in A$ with $0 < |w - c| < \delta$.

If $x, x' \in A$ with $0 < |x - c| < \delta$ and $0 < |x' - c| < \delta$, we have

$$|f(x) - f(x')| \leq |f(x) - L| + |f(x') - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(iii) $\Longrightarrow$ (ii) Suppose f satisfy (iii).

Pick arbitrary sequence $\{x_n\}$ with $x_n \in A \setminus \{c\}$ $\forall\, n \in \mathbb{N}$ and $\lim\limits_{n} x_n = c$.

Fixed any $\varepsilon > 0$, by assumption, we can find some $\delta > 0$, s.t.

$\forall\, x, x' \in A$ with $0 < |x - c| < \delta$ and $0 < |x' - c| < \delta$, we have $|f(x) - f(x')| < \varepsilon$. (*)

For this $\delta > 0$, by convergence and assumption of $\{x_n\}$, $\exists\, N \in \mathbb{N}$, s.t. $0 < |x_n - c| < \delta$ $\forall\, n \geq N$.

By (*), we have $|f(x_n) - f(x_m)| < \varepsilon$ $\forall\, n, m \geq N$.

Hence, $\{f(x_n)\}$ is Cauchy and so Convergent by Cauchy Convergent Theorem for Sequence.

(ii) $\Longrightarrow$ (i) Suppose f satisfy (ii).

Claim: $\lim\limits_{n} f(x_n)$ is SAME whenever $\{x_n\}$ is a sequence with $x_n \in A \setminus \{c\}$ $\forall\, n \in \mathbb{N}$ and $\lim\limits_{n} x_n = c$.

Proof Let $\{x_n\}, \{y_n\}$ be two sequences satisfying

$x_n, y_n \in A \setminus \{c\}$ $\forall\, n \in \mathbb{N}$ and $\lim_n x_n = c = \lim_n y_n$.

Suppose $\lim_n f(x_n) = L$ and $\lim_n f(y_n) = L'$ for some $L, L' \in \mathbb{R}$.

Now, we construct a new sequence $\{z_n\}$ by $z_{2n} = x_n$ and $z_{2n-1} = y_n$ for any $n \in \mathbb{N}$.

Then $z_n \in A \setminus \{c\}$ $\forall\, n \in \mathbb{N}$ and $\lim_n z_n = 0$. (I left this statement as exercise.)

Hence, $\lim_n f(z_n) = L''$ for some $L'' \in \mathbb{R}$.

Note that $\{f(x_n)\}, \{f(y_n)\}$ are subsequences of $\{f(z_n)\}$ and so we must have $L = L'' = L'$.

By the claim, $\exists\, L \in \mathbb{R}$, s.t. for any sequence $\{x_n\}$ with $x_n \in A \setminus \{c\}$ $\forall\, n \in \mathbb{N}$ and $\lim_n x_n = c$,

we have $\lim_n f(x_n) = L$. (**)

Suppose it were true that $\lim_{x \to c} f(x)$ does not exist. In particular, $\lim_{x \to c} f(x) \neq L$.

$\exists\, \varepsilon_0 > 0$, s.t. $\forall\, n \in \mathbb{N}$, $\exists\, a_n \in A$ with $0 < |a_n - c| < \frac{1}{n}$, s.t. $|f(a_n) - L| \geq \varepsilon_0$.

Note $\{a_n\}$ is a sequence with $a_n \in A \setminus \{c\}$ and $\lim_n a_n = c$ BUT $\lim_n f(a_n) \neq L$.

Contradiction with (**). Hence, $\lim_{x \to c} f(x) = L \in \mathbb{R}$.

4 Continuous Function

4.1 Basic Property

4.1.1 Definition

Let $f : A \to \mathbb{R}$, A non-empty subset of $\mathbb{R}$, let $c \in A$.

f is said to be continuous at c if

$\forall\, \varepsilon > 0,\ \exists\, \delta > 0$, s.t. $\forall x \in A$ with $|x - c| < \delta$, we have $\left|f(x) - f(c)\right| < \varepsilon$.

Also, f is said to be continuous on A if f is continuous at every $c \in A$.

Remark

(i) c must need to be in A, otherwise, $f(c)$ is NOT well-defined.

(ii) It is NOT necessary for c to be a cluster point of A.

(iii) If c is not a cluster point of A (we called it isolated point), then f is automatically continuous at c.

(iv) If c is a cluster point of A, then f is continuous at c is equivalent to

$$\lim_{\substack{x \to c \\ x \in A}} f(x) = f(c),$$

but in this course, please do NOT use this equivalent definition.

4.1.2 Property

If $f, g : A \to \mathbb{R}$ are continuous at $c \in A$, then fg is also continuous at c.

Proof

Suppose $f, g : A \to \mathbb{R}$ are continuous at $c \in A$.

Claim: g is locally bounded at 0. i.e. $\exists\, M > 0, \delta_1 > 0$, s.t. $\left|g(x)\right| < M\ \forall\, x \in A$ with $|x - c| < \delta_1$.

Proof Take $\varepsilon_0 = 1$, since g is continuous at c, $\exists\, \delta_1 > 0$, s.t.
$\left|g(x) - g(c)\right| < \varepsilon_0 = 1\ \forall\, x \in A$ with $|x - c| < \delta_1$.
That is, $f(x) < \text{Max}\left\{\left|g(c) + 1\right|, \left|g(c) - 1\right|\right\} =: M\ \forall\, x \in A$ with $|x - c| < \delta_1$.

Fixed any $\varepsilon > 0$, by f, g continuous at c, we can find

$\delta_2 > 0$, s.t. $\forall x \in A$ with $|x - c| < \delta_2$, we have $\left|f(x) - f(c)\right| < \dfrac{\varepsilon}{2M}$ and

$\delta_3 > 0$, s.t. $\forall x \in A$ with $|x - c| < \delta_3$, we have $\left|g(x) - g(c)\right| < \dfrac{\varepsilon}{2\left|f(c)\right| + 1}$.

Take $\delta = \text{Min}\,\{\delta_1, \delta_2, \delta_3\} > 0$, if $x \in A$ with $|x - c| < \delta$, we have

$$\begin{aligned}
\left|f(x)g(x) - f(c)g(c)\right| &\le \left|f(x) - f(c)\right|\left|g(x)\right| + \left|f(c)\right|\left|g(x) - g(c)\right| \\
&\le \frac{\varepsilon}{2M} \cdot M + \left|f(c)\right| \cdot \frac{\varepsilon}{2\left|f(c)\right| + 1} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, fg is also continuous at c.

4.1.3 Property

Let $A, B \subset \mathbb{R}$.

If $f : B \to \mathbb{R}, g : A \to B$ are continuous functions, then $f \circ g$ is also continuous on A.

Proof

Fixed any $\varepsilon > 0$, Fixed any $c \in A$, by continuity of f,

we can find $\eta > 0$, s.t. $\forall\ y \in B$ with $\left|y - g(c)\right| < \eta$, we have $\left|f(y) - f\left(g(c)\right)\right| < \varepsilon$. (*)

For this $\eta > 0$, by continuity of g,

we can find $\delta > 0$, s.t. $\forall\ x \in A$ with $|x - c| < \delta$, we have $\left|g(x) - g(c)\right| < \eta$.

Combine with (*), we know $\forall\ x \in A$ with $|x - c| < \delta$, we have $\left|f\left(g(x)\right) - f\left(g(c)\right)\right| < \varepsilon$.

Hence, $f \circ g$ is also continuous on A.

Question

Let $\varnothing \neq A \subset \mathbb{R}$, Let $f : \mathbb{R} \to \mathbb{R}$ be the distance function from A. That is,

$$f(x) := \operatorname{Inf}\left\{|x - a| : a \in A\right\}.$$

(a) Show $f(x) \leq |x - y| + f(y)$ for any $x, y \in \mathbb{R}$.

(b) Show f is continuous on $\mathbb{R}$.

(c) Let $c \notin A$. Show c is a cluster point of A if and only if $f(c) = 0$.

(d) Can we drop the assumption $c \notin A$ in part (c)?

Answer

(a) Pick any $x, y \in \mathbb{R}$, $a \in A$, by triangle inequality, $|x - a| \leq |x - y| + |y - a|$.

By taking infimum over $a \in A$ on both sides, since infimum preserves order, we have

$$f(x) \leq |x - y| + f(y).$$

(b) Fixed any $\varepsilon > 0$, $x \in \mathbb{R}$, take $\delta = \varepsilon > 0$. If $y \in \mathbb{R}$ with $|x - y| < \delta$, by (a), we have

$f(x) - f(y) \leq |x - y|$ and $f(y) - f(x) \leq |x - y|$, and so $\left|f(x) - f(y)\right| \leq |x - y| < \delta = \varepsilon$.

Hence, f is continuous at every point $x \in \mathbb{R}$. Hence, f is continuous on $\mathbb{R}$.

(c)($\Longrightarrow$) Suppose $c \notin A$ is a cluster point of A. Fixed any $\varepsilon > 0$,

We can find some $a \in A$, such that $0 \leq |c - a| < \varepsilon$.

By definition of Inf, $f(c) = 0$.

($\Longleftarrow$) Suppose $f(c) = 0$, $c \notin A$. Fixed any $\varepsilon > 0$, by definition of f (i.e. by definition of Inf),

we can find some $a \in A$, such that $|c - a| < \varepsilon$. Note that $a \neq c$ since $a \in A$ and $c \notin A$,

that is, $\forall\ \varepsilon > 0, \exists\ a \in A \setminus \{c\}$, such that $|c - a| < \varepsilon$.

Hence, c is a cluster point of A.

(d) NO. Consider the counter example $A = \{0\}$,

then $f(0) = 0$ but 0 is NOT a cluster point of A.

4.2 Uniform Continuity

4.2.1 Definition

Let $\varnothing \neq A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function.

f is said to be Uniformly Continuous on A if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \text{s.t. } \forall\, x, y \in A \text{ with } |x - y| < \delta,\ \text{ we have } \left|f(x) - f(y)\right| \leq \varepsilon.$$

Remark

(i) The uniform continuity of f is defined on some set but not a point.

(ii) If f is uniformly continuous on A, then f is continuous on A.

4.2.2 Example

(a) $f(x) = x$ is uniformly continuous on $\mathbb{R}$.

(b) $f(x) = x^2$ is uniformly continuous on $[a, b]$ for any $a, b \in \mathbb{R}$ with $a < b$.

However, $f(x) = x^2$ is NOT uniformly continuous on $\mathbb{R}$ but it is continuous on $\mathbb{R}$.

(c) $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[a, b]$ for any $a, b \in \mathbb{R}$ with $0 < a < b$.

However, $f(x) = \dfrac{1}{x}$ is NOT uniformly continuous on $(0, b]$

but it is continuous on $(0, b]$ for any $b > 0$.

4.2.3 Uniform Continuity Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function for some $a, b \in \mathbb{R}$ with $a < b$.

Then f is uniformly continuous on $[a, b]$ if and only if f is continuous on $[a, b]$.

4.2.4 Question

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$.

(a) If $\lim\limits_{x\to+\infty} f(x) = L \in \mathbb{R}$ and $\lim\limits_{x\to-\infty} f(x) = L' \in \mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.

(b) If f is periodic with period $p > 0$, that is

$$f(x + p) = f(x) \text{ for any } x \in \mathbb{R},$$

then f is uniformly continuous on $\mathbb{R}$.

Answer

(a) Fixed any $\varepsilon > 0$, by $\lim\limits_{x\to+\infty} f(x) = L \in \mathbb{R}$ and $\lim\limits_{x\to-\infty} f(x) = L' \in \mathbb{R}$,

$\exists\, M > 0$, s.t. $\left|f(x) - L\right| < \dfrac{\varepsilon}{4}\ \forall\, x \geq M$ (*) and

$\exists\, M' < 0$, s.t. $\left|f(x) - L'\right| < \dfrac{\varepsilon}{4}\ \forall\, x \leq M'$ (**).

Note that f is continuous on $[M', M]$,

and hence f is uniformly continuous on $[M', M]$ by Uniform Continuity Theorem.

Therefore, $\exists\, \delta' > 0$, s.t. $\forall\, x, y \in [M', M]$ with $|x - y| < \delta$, we have $\left|f(x) - f(y)\right| < \dfrac{\varepsilon}{2}$ (***).

Let $\delta := \text{Min}\{\delta', M - M'\} > 0$.

Now, pick any $x, y \in \mathbb{R}$ with $|x - y| < \delta$, WLOG, assume $x \leq y$, There are five cases:

(Case 1) Suppose $x, y \in [M', M]$, then by (***), $\left|f(x) - f(y)\right| < \frac{\varepsilon}{2} < \varepsilon$.

(Case 2) Suppose $x, y \leq M'$, then by (**), we have

$$\left|f(x) - f(y)\right| \leq \left|f(x) - L'\right| + \left|f(y) - L'\right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon.$$

(Case 3) Suppose $x, y \geq M$, then by (*), we have

$$\left|f(x) - f(y)\right| \leq \left|f(x) - L\right| + \left|f(y) - L\right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon.$$

(Case 4) Suppose $x \leq M' \leq y$, then $y < M$, then using (***) and case 2, we have

$$\left|f(x) - f(y)\right| \leq \left|f(x) - f(M')\right| + \left|f(M') - f(y)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(Case 5) Suppose $x \leq M \leq y$, then $x > M'$, then using (***) and case 3, we have

$$\left|f(x) - f(y)\right| \leq \left|f(x) - f(M)\right| + \left|f(M) - f(y)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In any cases, we must have $\left|f(x) - f(y)\right| < \varepsilon$.

Hence, f is uniformly continuous on $\mathbb{R}$.

(b) Fixed any $\varepsilon > 0$, note that f is continuous on $[0, p]$,

hence f is uniformly continuous on $[0, p]$ by Uniform Continuity Theorem.

Hence, $\exists\, \delta' > 0$, s.t. $\forall\, x, y \in [0, p]$ with $|x - y| < \delta'$, we have $\left|f(x) - f(y)\right| < \frac{\varepsilon}{2}$. (*)

Let $\delta := \text{Min}\{\delta', p\} > 0$.

Pick any $x, y \in \mathbb{R}$ with $|x - y| < \delta$, WLOG, assume $x \leq y$, by division algorithm,

$\exists!\, n, m \in \mathbb{Z}, s, t \in [0, p)$, s.t. $x = np + s$ and $y = mp + t$.

Note that $m \geq n$ and $-p < t - s < p$.

Note that $p \geq \delta > |x - y| = y - x = (m - n)p + (t - s) > (m - n - 1)p$.

Since $p > 0$, we have $0 \leq m - n < 2$, since $m, n \in \mathbb{Z}$, $m - n$ is either 0 or 1.

(Case 1) Suppose $m - n = 0$, that is $m = n$, so $|s - t| = |x - y| < \delta \leq \delta'$,
then by f is p–periodic and (*), we have

$$\left|f(x) - f(y)\right| = \left|f(np + s) - f(mp + s)\right| = \left|f(s) - f(t)\right| < \frac{\varepsilon}{2} < \varepsilon.$$

(Case 2) Suppose $m - n = 1$,
then $|p - s| = p - s \leq t + p - s = |t + p - s| = |x - y| < \delta \leq \delta'$,
and $\left|t - 0\right| = t \leq t + p - s = |t + p - s| = |x - y| < \delta \leq \delta'$,
then by f is p–periodic and (*), we have

$$\begin{aligned}\left|f(x) - f(y)\right| &\leq \left|f(np + s) + f(np + p)\right| + \left|f(np + p) + f(np + p + t)\right| \\ &= \left|f(s) - f(p)\right| + \left|f(0) - f(t)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

In any cases, $\left|f(x) - f(y)\right| < \varepsilon$.

Hence, f is uniformly continuous on $\mathbb{R}$.

4.3 Maximum Minimum Value Theorem

4.3.1 Main Statement

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$.

Then f attains an global maximum AND global minimum on $[a, b]$.

That is, $\exists\, x^*, x_* \in [a, b]$, s.t. $f(x_*) \leq f(x) \leq f(x^*)\ \forall x \in [a, b]$.

4.3.2 Question

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$.

(a) If $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = L \in \mathbb{R}$,
then f attains an global maximum OR global minimum on $\mathbb{R}$.

(b) With same assumption of (a),
could f attains both global maximum AND global minimum on $\mathbb{R}$?

(c) Could the assumption of (a) be replaced by $\lim_{x\to+\infty} f(x) = L \in \mathbb{R}$, $\lim_{x\to-\infty} f(x) = L' \in \mathbb{R}$?

answer

Let $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x) - L \; \forall \, x \in \mathbb{R}$.

Note that g is continuous of $\mathbb{R}$ with $\lim_{x\to+\infty} g(x) = \lim_{x\to-\infty} g(x) = 0$.

There are three cases:

(Case 1) Suppose $g(x) = 0 \; \forall \, x \in \mathbb{R}$,
that is g is a zero constant function,
then global maximum of g = global minimum of $g = 0$ (attains at everywhere).
Hence, global maximum of f = global minimum of $f = L$ (attains at everywhere).

(Case 2) Suppose $g(c) > 0$ for some $c \in \mathbb{R}$.
Take $\varepsilon_0 = \dfrac{g(c)}{2} > 0$, by $\lim_{x\to+\infty} g(x) = \lim_{x\to-\infty} g(x) = 0 \in \mathbb{R}$,
we can find $M' < 0$ and $M > 0$, such that $|g(x)| < \varepsilon_0 = \dfrac{g(c)}{2} \; \forall x \geq M$ or $x \leq M'$.
In particular, $g(x) \leq \dfrac{g(c)}{2} \; \forall x \geq M$ or $x \leq M'$. (*)
Also, we know $c \in [M', M]$ since $x = c$ does not satisfy $|g(x)| < \dfrac{g(c)}{2}$.
Note that g is continuous on $[M', M]$, by Maximum Minimum Value Theorem,
there exist some $x^* \in [M', M] \subset \mathbb{R}$, such that $g(x^*) \geq g(x) \; \forall \, x \in [M', M]$. (**)
If $x \geq M$ or $x \leq M'$, combine (*) and (**), we have

$$g(x) \leq \frac{g(c)}{2} < g(c) \leq g(x^*).$$

This means $g(x^*) \geq g(x) \; \forall \, x \in \mathbb{R}$,
that is $f(x^*) \geq f(x) \; \forall \, x \in \mathbb{R}$ by adding L on both sides.
Hence, f attain a global maximum at x^*.

(Case 3) Suppose $g(c) < 0$ for some $c \in \mathbb{R}$.
Then $-g(c) > 0$ for that $c \in \mathbb{R}$, apply **(case 2)** on $-g$,
there exist some $x_* \in \mathbb{R}$, such that $-g(x_*) \geq -g(x) \; \forall \, x \in \mathbb{R}$.
That is, $g(x_*) \leq g(x) \; \forall \, x \in \mathbb{R}$ and
hence, $f(x_*) \leq f(x) \; \forall \, x \in \mathbb{R}$ by adding L on both sides.

However, these f may not attain both global minimum and maximum.

Consider the counter example: $f(x) = \dfrac{1}{1 + x^2} \; \forall \, x \in \mathbb{R}$.

Note that f is well-defined continuous function on $\mathbb{R}$ (since $1 + x^2 > 0 \; \forall \, x \in \mathbb{R}$)

and $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.

Also, f attains a global maximum 1 at $x = 0$.

However, if f attained a global minimum at $x = c \in \mathbb{R}$, WLOG, assume $c > 0$, note that $f(c+1) < f(c)$ which is a contradiction.

Hence, f does NOT attain a global minimum.

If the limit of f as x tends to $\pm\infty$ is NOT same, the result may fail.

Consider the counter example: $f(x) = \begin{cases} 1 - \dfrac{1}{1+x^2}, & \text{if } x \geq 0 \\ \dfrac{1}{1+x^2} - 1, & \text{if } x < 0 \end{cases}.$

Note that f is continuous on $\mathbb{R}$. (please check it at least for $x = 0$ yourself!)

Also, $\lim_{x\to+\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$.

By same skill above, consider f is increasing on $\mathbb{R}$, (I left it as exercise.)

f does NOT attain ANY global maximum and minimum.

4.4 Intermediate Value Theorem

4.4.1 Main Statement

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$.

for any $k \in \mathbb{R}$ between $f(a)$ and $f(b)$, there exist $\xi \in [a, b]$, such that $f(\xi) = k$.

4.4.2 Question

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$.

If $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, then f is surjective.

Answer

Pick any $y \in \mathbb{R}$,

by $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, we can find $M' < 0$ and $M > 0$,

such that $f(x) > y \; \forall \; x \geq M$ and $f(x) < y \; \forall \; x \leq M'$.

In particular, $f(M) > y > f(M')$.

By Intermediate Value Theorem, we can find $x_0 \in (M', M) \subset \mathbb{R}$ such that $y = f(x_0)$.

That is, $\forall \; y \in \mathbb{R}, \; \exists \; x \in \mathbb{R}, \text{ s.t. } y = f(x)$.

Hence, $f : \mathbb{R} \to \mathbb{R}$ is surjective.